Coulomb Energy, Vortices, and Confinement

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We estimate the Coulomb energy of static quarks from a Monte Carlo calculation of the correlator of timelike link variables in Coulomb gauge. We find, in agreement with Cucchieri and Zwanziger, that this energy grows linearly with distance at large quark separations. The corresponding string tension, however, is several times greater than the accepted asymptotic string tension, indicating that a state containing only static sources, with no constituent gluons, is not the lowest energy flux tube state. The Coulomb energy is also measured on thermalized lattices with center vortices removed by the de Forcrand–D'Elia procedure. We find that when vortices are removed, the Coulomb string tension vanishes.

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I. INTRODUCTION

There is an old idea about confinement in Coulomb gauge which was originally put forward by Gribov [1], and which has been advocated in recent years by Zwanziger [2]. The idea is roughly as follows: For an SU(N) gauge theory fixed to "minimal" Coulomb gauge, the path integral is restricted to the region of configuration space in which the Faddeev–Popov operator

$$M^{ac} = -\partial_i D_i^{ac}(A) = -\nabla^2\delta^{ac} - \varepsilon^{abc}A_i^b\partial_i \tag{1.1}$$

has only positive eigenvalues. The boundary of this region in configuration space, beyond which M acquires zero or negative eigenvalues, is known as the "Gribov Horizon." In Coulomb gauge, the inverse of the Faddeev–Popov operator enters into the non-local part of the Hamiltonian

$$H = \frac{1}{2}\int d^3x\,(\vec{E}^{a,tr}\cdot\vec{E}^{a,tr} + \vec{B}^a\cdot\vec{B}^a) + \frac{1}{2}\int d^3x d^3y\,\rho^a(x)K^{ab}(x,y)\rho^b(y), \tag{1.2}$$

where ρ^a is the (matter plus gauge field) color charge density, $\vec{E}^{a,tr}$ is the transverse color electric field operator, and

$$K^{ab}(x,y) = \left[M^{-1}(-\nabla^2)M^{-1}\right]^{ab}_{x,y} \tag{1.3}$$

is the instantaneous Coulomb propagator. The expectation value of the non-local $\rho K\rho$ term in the Hamiltonian gives us the Coulomb energy. Now, since the dimension of configuration space is very large, it is reasonable that the bulk of configurations are located close to the horizon (just as the volume measure $r^{d-1}dr$ of a ball in d-dimensions is sharply peaked near the radius of the ball). Since it is the inverse of the M operator which appears in the Coulomb energy, it is possible that the near-zero eigenvalues of this operator will enhance the magnitude of the energy at large quark separations, possibly resulting in a confining potential at large distances. It may be, then, that the static quark potential is simply the Coulomb potential. In a diagrammatic analysis, the area-law falloff of large timelike Wilson loops in Coulomb gauge would be obtained by exponentiating instantaneous one-gluon exchange, going like

$$D_{00}(\mathbf{k},t) \sim \frac{A}{|\mathbf{k}|^4}\delta(t) \tag{1.4}$$

at small $|\mathbf{k}|$.

An objection that can be raised to this scenario is that if the confining potential is coming just from one-gluon exchange, it would be very hard to understand the origin of string-like behavior indicated by roughening [3] and the Lüscher term [4], for

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which there is now solid numerical evidence [5]. On the other hand, Zwanziger [6] has pointed out that the Coulomb energy of static sources is an upper bound on the static potential. This means that if the static potential is confining, then so is the Coulomb potential. It then seems economical to identify the two, as suggested by recent data on the Coulomb propagator reported by Cucchieri and Zwanziger [7].

In this article we calculate, by Monte Carlo techniques, the equal-times correlator of timelike link variables in Coulomb gauge. The logarithm of this quantity, in the continuum limit, is the Coulomb energy. We find, in agreement with Ref. [7], that the Coulomb energy rises linearly with distance at large quark separations. On the other hand, we also find that the slope is far too high to identify the Coulomb string tension with the usual asymptotic string tension. We will discuss the relevance of this result to the gluon chain proposal of Thorn and one of us (J.G.) [8], and in connection with some related comments of 't Hooft [9].

A second motivation of this article is to find out if there is a connection between the confining Coulomb potential and the center vortex confinement mechanism. We have therefore repeated our calculation of the Coulomb energy, in thermalized lattice configurations, with vortices removed by a procedure introduced by de Forcrand and D'Elia [10]. We find that vortex removal also removes the Coulomb string tension.

II. LINK CORRELATORS AND THE COULOMB ENERGY

We begin by defining the correlator, in Coulomb gauge, of two timelike Wilson lines

$$G(R,T) = \langle \mathrm{Tr}[L^\dagger(\mathbf{x},T)L(\mathbf{y},T)]\rangle, \tag{2.1}$$

where $R = |\mathbf{x}-\mathbf{y}|$ and

$$L(\mathbf{x},T) = P\exp\left[i\int_0^T dtA_0(\mathbf{x},t)\right]. \tag{2.2}$$

Wick rotation to Euclidean time is understood. This correlator represents the creation, at time $t=0$, of two static sources in a (global) color singlet state separated by a spatial distance R. This is a physical state in Coulomb gauge, denoted by Ψ_{qq}, and can be represented by massive quark/antiquark creation operators $\overline{q}(\mathbf{x})q(\mathbf{y})$ acting on the true vacuum state Ψ_0. The color sources propagate for a time T, and are then annihilated. From the existence of a transfer matrix we have

$$G(R,T) = \sum_n |c_n|^2 e^{-E_nT}, \tag{2.3}$$

where the sum is over all states $|n\rangle$ that have non-vanishing overlap c_n with Ψ_{qq}, and E_n is the energy of the n-th state above the vacuum energy.

The energy expectation value of the state Ψ_{qq}, above the vacuum energy, consists of an R-independent self-energy term E_{se}, plus an R-dependent potential. In this state, an R-dependent energy can only arise from the expectation value of the non-local Coulomb term in the Hamiltonian, so the R-dependent part of the energy is the Coulomb potential between static sources $V_{coul}(R)$. We have

$$\mathcal{E} = \left\langle\Psi_{qq}|H|\Psi_{qq}\right\rangle - \langle\Psi_0|H|\Psi_0\rangle = V_{coul}(R) + E_{se}. \tag{2.4}$$

Defining

$$\mathcal{V}(R,T) = -\frac{d}{dT}\log[G(R,T)] \tag{2.5}$$

it is easy to see that [11]

$$\mathcal{E} = \lim_{T\to 0}\mathcal{V}(R,T), \tag{2.6}$$

while

$$E_{min} = \lim_{T\to\infty}\mathcal{V}(R,T) = V(R) + E'_{se}, \tag{2.7}$$

where E_{min} is the minimum energy, above the vacuum energy, of the quark-antiquark system, $V(R)$ is the usual static quark potential, and E'_{se} is an R-independent self-energy. The minimum energy state, which dominates the sum over states in Eq. (2.3), is expected to represent a flux tube in its ground state, stretching between the static sources.

The self-energies E_{se} and E'_{se} are regulated when the gauge theory is formulated on the lattice. If the static quark potential is confining, then these self-energies must be negligible, at sufficiently large R, compared to the static potential. Then, from the fact that $E_{min} \leq \mathcal{E}$, if follows that

$$V(R) \leq V_{coul}(R), \tag{2.8}$$

as first pointed out by Zwanziger [6]. This means that if the static quark potential is confining, the instantaneous Coulomb potential is also confining.

With a Euclidean lattice regularization, we define

$$L(\mathbf{x},T) = U_0(\mathbf{x},a)U_0(\mathbf{x},2a)...U_0(\mathbf{x},T) \tag{2.9}$$

as the appropriate ordered product of link variables, and

$$V(R,T) = \frac{1}{a}\log\left[\frac{G(R,T)}{G(R,T+a)}\right], \tag{2.10}$$

where a is the lattice spacing. The identity (2.6) then holds only in the continuum limit, i.e.

$$\lim_{\beta\to\infty} V(R,0) = V_{coul}(R) + \text{const.}, \tag{2.11}$$

while we still have, at any β,

$$\lim_{T\to\infty} V(R,T) = V(R) + \text{const.}, \tag{2.12}$$

where the constants are self-energies, and

$$V(R,0) = -\frac{1}{a}\log[G(R,a)]. \tag{2.13}$$

By calculating $V(R,0)$ via lattice Monte Carlo, we may address several questions:

1. Does $V(R,0)$ (and hence the Coulomb potential) increase linearly with R at large β?
2. If so, does the associated Coulomb string tension σ_{coul} match the usual asymptotic string tension σ of the static quark potential?
3. If center vortices are removed from thermalized lattice configurations, what happens to the measured Coulomb potential?

III. NUMERICAL RESULTS

From here on we will work in lattice units, with $a = 1$. Lattice sizes used in numerical simulations are $16^4, 16^4, 20^4$, and 24^4 at $\beta = 2.2, 2.3, 2.4$, and 2.5 respectively. Data points are derived from 500 configurations separated by 50 sweeps at each β.

First, as a check of Eq. (2.12), it is useful to verify that the string tension $\sigma(T)$ extracted from $V(R,T)$ approaches the accepted asymptotic string tension at large T, where the link correlator $G(R,T)$ is computed in lattice Coulomb gauge (implemented by the over-relaxation technique). String tensions $\sigma(T)$ are extracted from a fit of $V(R,T)$ to the form

$$V(R,T) = c(T) - \frac{\pi}{12R} + \sigma(T)R \tag{3.1}$$

in the range $R_{min} \leq R \leq R_{max}$, where we have used $R_{min} = 3,3,4,4$, and $R_{max} = 5,6,8,10$ at $\beta = 2.2, 2.3, 2.4, 2.5$ respectively. The results for $\sigma(T)$ vs. T at $\beta = 2.3 - 2.5$ are shown in Fig. 1. The data does appear to converge towards the accepted asymptotic string tension as T increases. The data for $V(R,4)$ at $\beta = 2.5$ is shown in Fig. 2 (data points marked “without vortices” will be explained below).[1] In this case $\sigma(4) = 0.0402(2)$, which can be compared to the accepted asymptotic string tension $\sigma = 0.0350(4)$ at this coupling [12]. We note that timelike Wilson line correlators, in a physical gauge, have been

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[1] In order to avoid multiply overlapping, overcrowded symbols, not all available data points are displayed in Figs. 2 and 3.

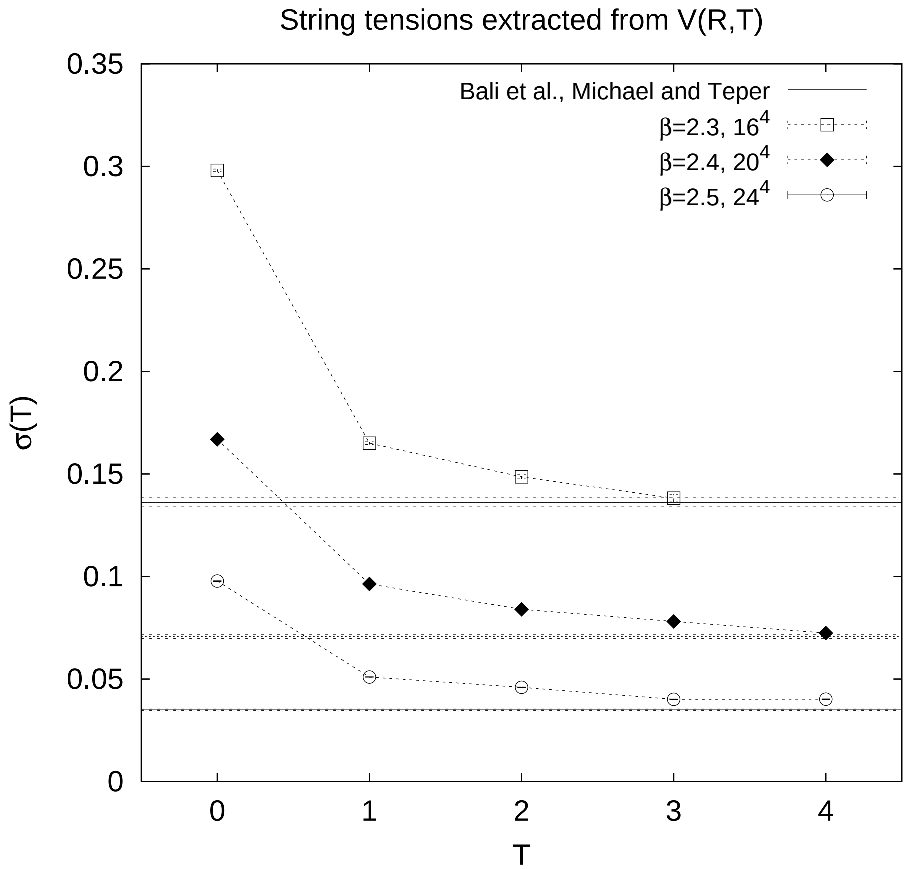

FIG. 1: Falloff of $\sigma(T)$ with increasing T at $\beta = 2.3, 2.4, 2.5$. Solid lines indicate the accepted values of the asymptotic string tension at each β value, with dashed lines indicating the error bars.

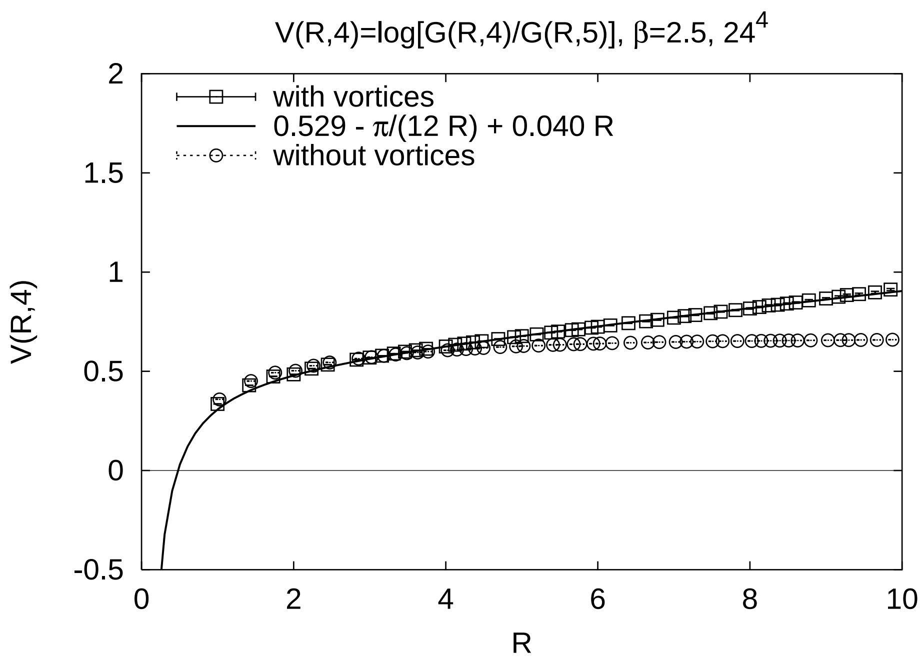

FIG. 2: $V(R,4)$ at $\beta = 2.5$. The "without vortices" data points are obtained on lattices with vortices removed by the de Forcrand–D'Elia procedure [10].

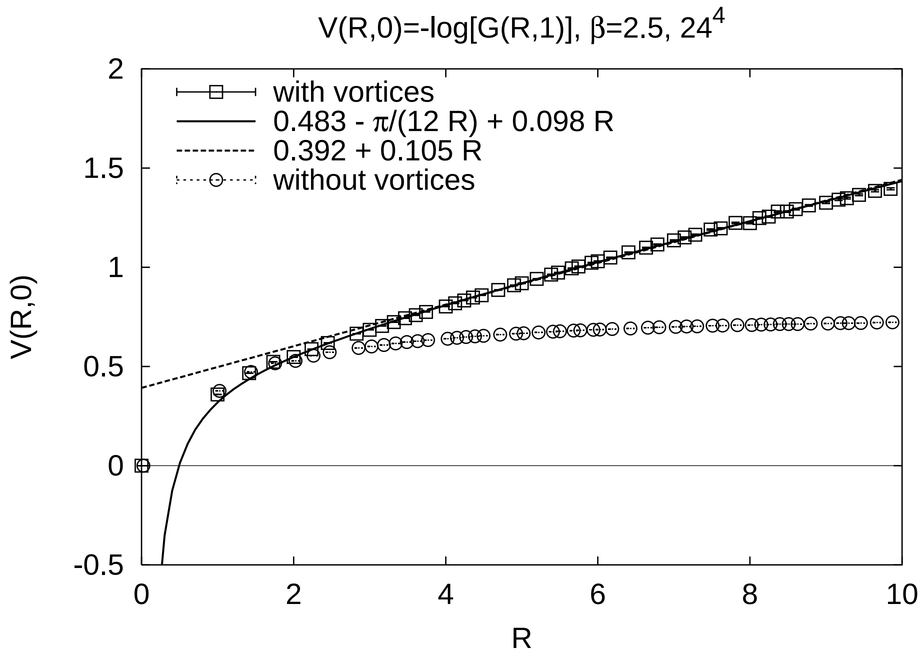

FIG. 3: $V(R,0)$ at $\beta = 2.5$, which is related [Eq. (2.11)] to the Coulomb energy. The solid (dashed) line is a fit to a linear potential with (without) the Lüscher term.

used previously to compute static potentials, most recently by de Forcrand and Philipsen [13] in connection with adjoint string-breaking. The static ground-state potential is obtained from the asymptotic correlators at large T, but in connection with the Coulomb potential we are interested in the opposite, small T limit.

As already explained, the Coulomb potential is obtained at large β from $V(R,0)$. At all four values of β that we have used in our simulations, $V(R,0)$ is clearly a linear function of R at large R, and we see no reason that this behavior would change as β is increased. This means that the instantaneous Coulomb potential is also linear at large R, and the first question posed at the end of Section 2 can be answered affirmatively, in agreement with Cucchieri and Zwanziger [7]. Our data for $V(R,0)$ at $\beta = 2.5$ is shown in Fig. 3.

On the other hand, there is no indication that $\sigma(0) \approx \sigma(\infty)$ at large β, which is what is required if the Coulomb string tension σ_{coul} would agree with the usual asymptotic string tension σ. If anything, there is the opposite tendency as β increases. In Fig. 4 we plot the ratio $\sigma(0)/\sigma$ as a function of β. We have obtained $\sigma(0)$ from two different fits, with and without the Lüscher term $-\pi/12R$. It makes sense to include the Lüscher term at large T, but it is a little hard to see how such a term, derived from string-like fluctuations, would originate due to instantaneous one-gluon exchange, which is the origin of the Coulomb force. So we have also extracted $\sigma(0)$ from a fit in which the Lüscher term is dropped in Eq. (3.1). In any case the ratios $\sigma(0)/\sigma$, extracted from fits with and without the Lüscher term, are not much different, and both results are shown in Fig. 4. These ratios show a tendency to increase with β, and we do not really see convergence to a finite value, at least in this range of β, and certainly no evidence that the ratio converges to unity. If the rise in $\sigma(0)/\sigma$ is monotonic in β, then assuming the ratio converges at all, it is unlikely to be less than $\sigma(0)/\sigma = 3$. The data therefore appears to give a negative answer to the second question posed at the close of the previous section. Our results are not compatible with $\sigma_{coul} \approx \sigma$, and we differ in this respect from the conclusions of Ref. [7].

A. Vortex Removal

The idea that confinement is entirely due to the Coulomb potential, arising from the instantaneous part of the A_0A_0 propagator, has certain problematic features. Apart from the issue of the Lüscher term, it is not entirely clear how the Coulomb propagator would explain the string tension of spacelike Wilson loops, which are constructed from the transverse gluon field.[2] An alternative

[2] We thank D. Zwanziger for this comment.

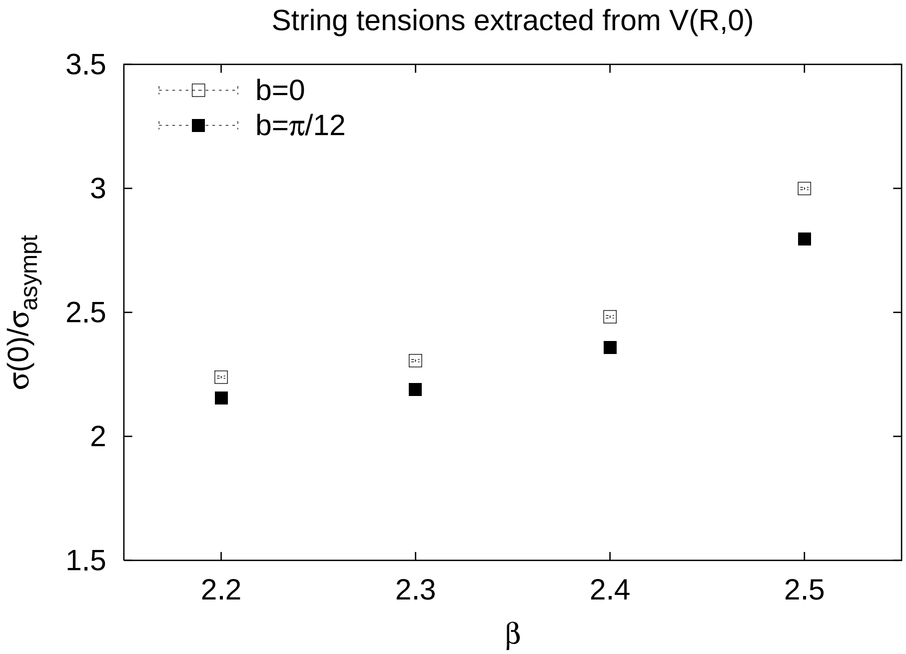

FIG. 4: The ratio $\sigma(0)/\sigma$, at various β, from fits which include ($b=\pi/12$) or do not include ($b=0$) the Lüscher term. This ratio should equal the ratio of the Coulomb to asymptotic string tension in the $\beta\rightarrow\infty$ limit.

explanation of the confining force in terms of center vortices has been extensively investigated (cf. the review in Ref. [14] and references therein), and this mechanism can be invoked to obtain an area law falloff for spacelike loops as well as timelike loops. It is interesting, then, to ask if there is some relation between the linear confining Coulomb potential found in Coulomb gauge, and the effects of center vortices identified by center projection in an adjoint gauge.

To study this question, we adopt the "vortex removal" method devised by de Forcrand and D'Elia [10]. In SU(2) lattice gauge theory, a thermalized lattice is fixed to maximal center gauge, and then the links are modified according to the rule

$$U_\mu(x)\rightarrow U'_\mu(x)=\text{signTr}[U_\mu(x)]\,U_\mu(x). \tag{3.2}$$

The modified configuration is still in maximal center gauge, but has no vortices upon center projection. The procedure can be visualized as placing thin center vortices in the middle of thick center vortices; the effect of the two types of vortices on large loops cancel out. It is well known that the string tension of Wilson loops in the modified configuration vanishes [10]. For present purposes we gauge fix the modified $U'_\mu(x)$ configuration to Coulomb gauge, and compute $V(R,T)$ as before. The result, labelled "without vortices" in Figs. 2 and 3, is that the string tension $\sigma(T)$ vanishes at every T and every β. There is no Coulomb string tension, and no asymptotic string tension, when center vortices are removed.[3]

IV. COULOMB ENERGY AND THE GLUON CHAIN MODEL

We have alluded several times to the string-like behavior of the QCD flux tube, which manifests itself both in the phenomenon of "roughening," i.e. the logarithmic growth of flux tube thickness with quark separation, and also by the presence of the Lüscher term in the static potential. It is not obvious how this string-like behavior would be obtained from instantaneous one-gluon exchange, even given that such an exchange generates a linear confining potential. Our data also indicates that the purely Coulombic force, at long range, may be several times greater than the actual asymptotic force between static quarks. We would like to suggest that the two issues are related: the Coulombic force is lowered to the true asymptotic force by constituent gluons in the QCD flux tube, and the fluctuations of these gluons in transverse directions accounts for the string-like phenomena. The general picture is known as the "gluon chain model," advocated by Thorn and one of us (J.G.) in Ref. [8].

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[3] The effect of vortex removal on Landau gauge propagators, and its possible implications for confinement, has been investigated by K. Langfeld *et al.* in Ref. [15].

The gluon chain model is motivated by the fact that a time-slice of a high order planar diagram for a Wilson loop reveals a sequence of gluons, with each gluon interacting only with its nearest neighbors in the diagram. This suggests that the QCD string might be regarded, in a physical gauge, as a “chain” of constituent gluons, with each gluon held in place by attraction to its two nearest neighbors in the chain. The linear potential in this model comes about in the following way: As heavy quarks separate, we expect that the Coulombic interaction energy increases rapidly due to the running coupling. Eventually it becomes energetically favorable to insert a gluon between the quarks, to reduce the effective color charge separation. As the quarks continue to separate, the process repeats, and we end up with a chain of gluons. The average gluon separation d along the axis joining the quarks is fixed, regardless of the quark separation R, and the total energy of the chain is the energy per gluon times the number $N = R/d$ of gluons in the chain, i.e.

$$E_{chain} \approx NE_{gluon} = \frac{E_{gluon}}{d} R = \sigma R, \tag{4.1}$$

where E_{gluon} is the (kinetic+interaction) energy per gluon, and $\sigma = E_{gluon}/d$ is the asymptotic string tension. In this picture, the linear growth in the number of gluons in the chain is the origin of the linear potential. A typical gluon-chain state would have the form

$$\begin{aligned} \Psi_{chain}[A] &= \overline{q}^{a_1}(\mathbf{x}) \Big\{ \int d\mathbf{x}_1 d\mathbf{x}_2 ... d\mathbf{x}_N \ \psi_{\mu_1\mu_2...\mu_N}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) \\ & \qquad A^{a_1a_2}_{\mu_1}(\mathbf{x}_1) A^{a_2a_3}_{\mu_2}(\mathbf{x}_2) ... A^{a_Na_{N+1}}_{\mu_N}(\mathbf{x}_N) \Big\} q^{a_{N+1}}(\mathbf{y}) \Psi_0[A] \end{aligned} \tag{4.2}$$

where ψ is a “string-bit” wavefunction correlating positions of the N constituent gluons in the chain, and is to be determined by minimizing the energy of the chain. This minimization only involves the interaction of neighboring gluons separated by an average distance d, rather than the direct interaction of quarks separated by a very large distance R. It was shown in Ref. [8], on the basis of a simplified quantum-mechanical model, that gluon-chain states can plausibly account for the observed string-like behavior of the QCD flux tube.

However, as pointed out last year by ’t Hooft [9], there are certainly states in the Fock space, containing the static sources, which are not gluon chain states. If the gluon chain scenario has any validity, then it must be that the interaction energy of these non-chain states is also confining, but of much higher energy than the gluon chain states.

This is where the numerical result of the last section, which indicates that $\sigma_{coul} > \sigma$, becomes relevant. The simplest “no-chain” state is a state with no constituent gluons at all, i.e.

$$\Psi_{qq} = \overline{q}^a(\mathbf{x}) q^a(\mathbf{y}) \Psi_0. \tag{4.3}$$

The R-dependent part of the energy expectation value of this state is precisely the Coulomb energy. If it is true, as our data suggests, that (i) the Coulomb potential is linearly confining; and (ii) the Coulomb string tension σ_{coul} is greater than the string tension σ of the usual static potential, then we find that this simplest no-chain state is indeed confining, and of higher energy than the lowest energy flux tube state, which we suggest has the form of a gluon chain.

V. CONCLUSIONS

Our numerical results, extracted from the correlators of timelike Wilson loops in Coulomb gauge, are relevant to several ideas about confinement. First of all, we have found that the Coulomb energy grows linearly with quark separation R at large R, in agreement with a result long maintained by Cucchieri and Zwanziger [7, 16]. We also find that when center vortices are removed from lattice configurations by the de Forcrand–D’Elia procedure, the Coulomb string tension σ_{coul} drops to zero. This finding lends further support to the contention that center vortices are crucial to the confinement property.

Finally, our data indicates that σ_{coul} is several times larger than the asymptotic string tension; a result which is entirely consistent with the inequality $V(R) \le V_{coul}(R)$ at large distances, and which bears on the validity of the gluon chain model. In order to have $\sigma_{coul} \approx \sigma$, it would be necessary that the string tension $\sigma(T)$, extracted from correlators of timelike lines of length T and $T+1$, should be almost independent of T, since $\sigma_{coul} = \sigma(0)$ at large β, and $\sigma = \sigma(\infty)$. This is not what is found; instead $\sigma(T)$ drops off sharply with T near $T = 0$. If this result holds at still larger values of β, then we must conclude that σ_{coul} is substantially higher than the usual asymptotic string tension.[4]

Assuming that in fact σ_{coul} is greater than σ, it follows that the simplest state (Ψ_{qq}) containing static sources but no constituent gluons is overconfining, and therefore has a negligible overlap with the true QCD flux tube state at very large quark separations.

[4] We note however that the authors of Ref. [7], who use inversion of the Faddeev–Popov operator to compute the Coulomb string tension, reach a somewhat different conclusion.

A lower interaction energy must be obtainable by operating on the vacuum with some arrangement of gluon operators, and the gluon chain model is a specific proposal for that arrangement. While $\sigma_{coul} > \sigma$ does not necessarily imply that the gluon chain proposal is right, the result $\sigma_{coul} \approx \sigma$ would have been a strong indication that the proposal is wrong.

Acknowledgments

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